

# THE REGULARITY OF PROJECTION OPERATORS AND SOLUTION OPERATORS TO $\bar{\partial}$ ON WEAKLY PSEUDOCONVEX DOMAINS

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ABSTRACT. We relate the existence and regularity of a solution operator to  $\bar{\partial}$  on smoothly bounded pseudoconvex domains to the existence and regularity of a projection operator onto the kernel of  $\bar{\partial}$ .

## 1. INTRODUCTION

In this article we show that the existence of a solution operator to the  $\bar{\partial}$ -equation satisfying estimates on Sobolev spaces of smoothly bounded pseudoconvex domains can be deduced from the existence of a continuous projection operator. We let  $\Omega \subset \mathbb{C}^n$  be a smoothly bounded pseudoconvex domain. We denote the Sobolev spaces by  $W_{(p,q)}^s(\Omega)$ , the space of  $(p,q)$ -forms whose derivatives of order  $\leq s$  (of its coefficients) are in  $L_{(p,q)}^2(\Omega)$ . Let  $D_1, \dots, D_{2n}$  be an orthonormal frame of  $2n$  vector fields. The norm  $\|\cdot\|_s$  attached to  $W_{(p,q)}^s(\Omega)$  is given by

$$\|f\|_s = \sum_{k=0}^s \sum_{j=1}^{2n} \|D_j^k f\|_{L_{(p,q)}^2(\Omega)}.$$

We look at the relation between the existence of a projection operator,  $\mathbf{T}_{(p,q)} : L_{(p,q)}^2(\Omega) \rightarrow L_{(p,q)}^2(\Omega) \cap \ker(\bar{\partial})$ , which is also a linear operator between Sobolev spaces, so that  $\mathbf{T}_{(p,q)} : W_{(p,q)}^s(\Omega) \rightarrow W_{(p,q)}^s(\Omega)$  for all  $s \geq 0$  and the existence of a solution operator,  $\mathbf{S}_{(p,q+1)} : L_{(p,q+1)}^2(\Omega) \cap \ker(\bar{\partial}) \rightarrow L_{(p,q)}^2(\Omega)$  such that  $\bar{\partial}\mathbf{S}_{(p,q+1)}f = f$  and  $\|\mathbf{S}_{(p,q+1)}f\|_s \lesssim \|f\|_s$  for all  $s \geq 0$ , where the constant in the inequality is independent of  $f$ . Our Main Theorem in this regard is

**Main Theorem.** *Let  $\Omega \subset \mathbb{C}^n$  be a smoothly bounded pseudoconvex domain. For  $0 \leq p \leq n$  and  $0 \leq q \leq n-1$ , suppose there exists a projection operator  $\mathbf{T}_{(p,q)} : L_{(p,q)}^2(\Omega) \rightarrow L_{(p,q)}^2(\Omega) \cap \ker(\bar{\partial})$ , with the property that  $\mathbf{T}_{(p,q)}$  is continuous as an operator*

$$\mathbf{T}_{(p,q)} : W_{(p,q)}^s(\Omega) \rightarrow W_{(p,q)}^s(\Omega) \quad \forall s \geq 0.$$

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Then there exists a solution operator,  $\mathbf{S}_{(p,q+1)}$ , such that  $\bar{\partial} \circ \mathbf{S}_{(p,q+1)} = I$  on  $\bar{\partial}$ -closed  $(p, q+1)$ -forms which is continuous as an operator

$$\mathbf{S}_{(p,q+1)} : W_{(p,q+1)}^s(\Omega) \cap \ker(\bar{\partial}) \rightarrow W_{(p,q)}^s(\Omega) \quad \forall s \geq 0.$$

Naively, one could also obtain estimates for a projection operator given a solution operator: given  $\mathbf{S}_{(p,q+1)}$ , then we could form the projection  $\mathbf{T}_{(p,q)} = I - \mathbf{S}_{(p,q+1)} \circ \bar{\partial}$ . Due to the  $\bar{\partial}$ -operator, however, there is a loss of one derivative in the reverse direction of the theorem. One would need estimates for the composite operator  $\mathbf{S}_{(p,q+1)} \circ \bar{\partial}$  to obtain a theorem with the reverse implication. Such an equivalence between the regularity of the Bergman projection and the  $\bar{\partial}$ -Neumann operator (and through it the regularity of the canonical solution to the  $\bar{\partial}$ -problem) was proved in [2]. Thus, if the Bergman projection  $P_{(p,q)}$  on  $(p, q)$ -forms satisfies the hypothesis of the theorem, we can take  $\mathbf{S}_{(p,q+1)} = \mathbf{C}_{(p,q+1)}$ , the canonical solution operator, or the operator giving the solution of minimal  $L^2$ -norm. However, we note that the Bergman projection does not always hold the desired properties of the projection operators in the Main Theorem. It is a result of Barrett that there exist smoothly bounded pseudoconvex domains on which the Bergman projection fails to map  $W^s(\Omega)$  to itself for large enough  $s$  [1]. Furthermore, the solution operators to the  $\bar{\partial}$ -equation on weakly pseudoconvex domains of Kohn [3] do not satisfy the desired properties of the solution operators of the Main Theorem. The solution operator of Kohn is only shown to give regularity up to a fixed level; for a fixed  $s$  a solution operator,  $K_{s,q+1}$ , can be constructed such that for a  $\bar{\partial}$ -closed  $(0, q+1)$ -form,  $f$  we have  $\bar{\partial} K_{s,q+1} f = f$  and  $K_{s,q+1} : W_{(0,q+1)}^k(\Omega) \rightarrow W^k(\Omega)$  for  $k \leq s$ . We are interested in the estimates holding simultaneously for all  $s$ .

The Main Theorem is proved by construction. Given a projection, a solution operator is constructed. Thus, the Main Theorem suggests one method of obtaining (regular) solution operators to  $\bar{\partial}$ . Namely, if one can find a projection operator which preserves Sobolev spaces, then a solution operator to the  $\bar{\partial}$ -equation can be found with no loss of derivatives.

## 2. PROOF OF THE MAIN THEOREM

It suffices to work with  $(0, q)$ -forms, and we simply drop the  $p$  component in the indices, e.g.  $W_q^s(\Omega) := W_{(0,q)}^s(\Omega)$ , etc.

Let  $\mathbf{C}_{q+1}$  be the canonical solution operator (i.e. the operator which gives the solution of minimal  $L^2$ -norm) to the  $\bar{\partial}$ -equation, and let  $\mathbf{P}_q$  denote the Bergman projection. We note the relation

$$(2.1) \quad \mathbf{P}_q = \mathbf{I} - \mathbf{C}_{q+1} \circ \bar{\partial}.$$

We also denote by  $K_{s,q+1}$  the solution of Kohn which maps  $W_{q+1}^s(\Omega)$  to  $W_q^s(\Omega)$ .

We show how to obtain a solution operator  $\mathbf{S}_{q+1}$  by combining the operators  $\mathbf{C}_{q+1}$ ,  $\mathbf{P}_q$ , and  $\mathbf{T}_q$ . We thus assume the existence of  $\mathbf{T}_q$  as in the Main Theorem, and we define

$$(2.2) \quad \mathbf{S}_{q+1} = \mathbf{C}_{q+1} + (\mathbf{P}_q - \mathbf{T}_q) \circ K_{0,q},$$

restricted to  $\ker(\bar{\partial})$ . We note that for  $s_1, s_2 \geq 0$ ,  $K_{s_1, q+1} - K_{s_2, q+1}$  maps  $\bar{\partial}$ -closed  $(0, q+1)$ -forms onto  $\bar{\partial}$ -closed  $(0, q)$ -forms. And, as both  $\mathbf{P}_q$  and  $\mathbf{T}_q$  reproduce  $\bar{\partial}$ -closed  $(0, q)$ -forms, we have

$$(\mathbf{P}_q - \mathbf{T}_q) \circ K_{s_1, q+1} \equiv (\mathbf{P}_q - \mathbf{T}_q) \circ K_{s_2, q+1}$$

on  $W_{q+1}^{\max\{s_1, s_2\}}(\Omega) \cap \ker(\bar{\partial})$ . We thus conclude that with  $\mathbf{S}_{q+1}$  defined as in (2.2) for any  $s \geq 0$ , we also have

$$\mathbf{S}_{q+1} = \mathbf{C}_{q+1} + (\mathbf{P}_q - \mathbf{T}_q) \circ K_{s, q},$$

on  $W_q^\infty(\Omega) \cap \ker(\bar{\partial})$ .

Because  $\bar{\partial} \circ (\mathbf{P}_q - \mathbf{T}_q) \equiv 0$ , for  $f \in L_{q+1}^2(\Omega)$  a  $\bar{\partial}$ -closed form we have

$$\begin{aligned} \bar{\partial} \mathbf{S}_{q+1} f &= \bar{\partial} \mathbf{C}_{q+1} f + \bar{\partial} (\mathbf{P}_q - \mathbf{T}_q) K_{0, q} f \\ &= f. \end{aligned}$$

We finish the proof of the Main Theorem by showing that for all  $s \geq 0$  we have  $\|\mathbf{S}_{q+1} f\|_s \lesssim \|f\|_s$  if  $f \in W_{q+1}^s(\Omega) \cap \ker(\bar{\partial})$ . With  $f \in W_{q+1}^s(\Omega) \cap \ker(\bar{\partial})$  we have

$$\begin{aligned} \mathbf{S}_{q+1} f &= \mathbf{C}_{q+1} f + (\mathbf{P}_q - \mathbf{T}_q) K_{0, q} f \\ &= \mathbf{C}_{q+1} \bar{\partial} K_{s, q} f + (\mathbf{P}_q - \mathbf{T}_q) K_{s, q} f \\ &= K_{s, q} f - \mathbf{T}_q K_{s, q} f, \end{aligned}$$

by (2.1). By assumption we have that  $\mathbf{T}_q$  preserves  $W_q^s(\Omega)$ , and we conclude

$$\begin{aligned} \|\mathbf{S}_{q+1} f\|_s &\lesssim \|K_{s, q+1} f\|_s \\ &\lesssim \|f\|_s. \end{aligned}$$

This finishes the proof of the Main Theorem.

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